RINGS AND MODULES CHARACTERIZED BY OPPOSITES OF ABSOLUTE PURITY

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- An opposite notion of relative injectivity which is introduced by (Aydoğdu and López-Permouth), studied by many authors.
- A module M is said to be A-subinjective if for every extension B of A any homomorphism $\varphi : A \to M$ can be extended to a homomorphism $\phi : B \to M$ (see, Aydoğdu and López-Permouth). It is easy to see that M is injective if and only if M is A-subinjective for each module A.

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- The idea and notion of subinjectivity can be used in order to study opposites of some other homological objects such as, absolutely pure and flat modules.
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- The purpose of this talk is to mention the study of an alternative perspective on the analysis of the absolute purity of a module.

Proposition

The following statements are equivalent for a right module N.

(1) *N is absolutely pure.*

(2) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for each finitely presented left module M.*

(3) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for each left module M.*

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Definition

Given a right module M and a left module N, *M is absolutely N-pure* if for every right module K with $M \leq K$ the map $i \otimes 1_N : M \otimes N \to K \otimes N$ is a monomorphism, where $i : M \to K$ is the inclusion map and 1_N is the identity map on N. The *subpurity domain* of a module M_R, $\mathcal{S}(M)$, is defined to be the collection of all modules ${}_RN$ such that M is absolutely N-pure.

- A right module M is absolutely pure if and only if $\mathcal{S}(M) = R - MOD$.
- $\mathcal{S}(M)$ consists of the class of left flat modules.
- A right R-module M is called *test for flatness by subpurity (t.f.b.s.)* if $\mathcal{S}(M_R)$ consists of only flat left R-modules.

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Example

The ring of integers $\mathbb{Z}$ is t.f.b.s.

Proposition

$\bigcap_{M \in MOD-R} \mathcal{S}(M) = \{N \in R - MOD | N \textit{ is flat}\}.$

Proposition

Every ring has a t.f.b.s. module.

Proposition

The following statements are equivalent for a ring R.

(1) *R is von Neumann regular.*

(2) *Every right R-module is t.f.b.s.*

(3) *There exists a right absolutely pure t.f.b.s. R-module.*

In order to investigate when the ring is t.f.b.s. as a right module over itself, we need the following definition.

Definition

A ring R is called right S-ring if every finitely generated flat right ideal is projective.

Theorem

A ring R is right t.f.b.s. and a right S-ring if and only if R is right semihereditary

Corollary

A commutative domain is Prüfer if and only if it is t.f.b.s.

Definition

A module A is said to be a *test module for injectivity by subinjectivity (or t.i.b.s)* if whenever a module M is A-subinjective implies M is injective (see, Alizade, Büyükaşık and Er).

Proposition

If N is right t.i.b.s., then N is right t.f.b.s.

There are t.f.b.s. modules which are not t.i.b.s.

Example

Every semihereditary ring is a t.f.b.s. as a right module over itself. On the other hand, R_R is t.i.b.s. if and only if R is right hereditary and right Noetherian (see, Alizade, Büyükaşık, and Er).

In searching the converse of above Proposition, we have the following.

Proposition

Let R be a right Noetherian ring. If M is a t.f.b.s. right R-module and $E(M)$ is finitely generated, then M is right t.i.b.s.

Proposition

The following are equivalent for a ring R.

(1) *R_R is t.f.b.s. and Noetherian.*

(2) *R_R is t.i.b.s.*

Outline

Theorem

Let R be a right Noetherian ring. Every simple right module is t.f.b.s. or absolutely-pure (=injective) if and only if

(1) *R is a right V-ring; or*

(2) *$R \cong A \times B$, where A is right Artinian with a unique non-injective simple right module and $Soc(A_A)$ is homogeneous and B is semisimple.*

Proposition

Let R be a right Noetherian ring. Every simple right module is t.f.b.s. or absolutely pure if and only if every simple right module is t.i.b.s. or injective.

Proposition

Let R be an arbitrary ring. Suppose that every simple module is t.i.b.s. or injective. Then R is a right V-ring or right Noetherian.

Corollary

The following are equivalent for a ring R.

(1) *Every simple module is t.i.b.s. or injective.*

(2) (i) *R is a right V-ring, or*
(ii) *$R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $Soc(A_A)$ is homogeneous and B is semisimple.*

Outline

Theorem

Let R be a right Noetherian ring. Suppose that every right R-module is t.f.b.s. or absolutely pure. Then $R \cong A \times B$, where B is semisimple, and

(1) *A is right hereditary right Artinian serial with homogeneous socle, $J(A)^2 = 0$ and A has a unique noninjective simple right A-module, or;*

(2) *A is a QF-ring that is isomorphic to a matrix ring over a local ring, or;*

(3) *A is right SI with $Soc(A_A) = 0$.*

Outline

Proposition

The following are equivalent for a commutative domain R.

(1) *R is Prüfer.*

(2) *R is t.f.b.s.*

(3) *Every nonzero finitely generated ideal is t.f.b.s.*

(4) *A finitely generated R-module M is t.f.b.s. when $Hom(M, R) \neq 0$.*

Now we shall give a characterization of t.f.b.s. modules over commutative hereditary Noetherian rings. We begin with the following.

Theorem

Let R be a commutative hereditary Noetherian ring and F a flat module. Then F is t.f.b.s. if and only if $Hom(F, S) \neq 0$ for each singular simple R-module S.

Theorem

Let R be a commutative hereditary Noetherian ring and N be an R-module. The following are equivalent.

(1) *N is t.f.b.s.*

(2) *$N/Z(N)$ is t.f.b.s.*

(3) *$Hom(N/Z(N), S) \neq 0$ for every singular simple R-module S.*

(4) *$N/Z(N) \otimes S \neq 0$ for every singular simple R-module S.*

Corollary

Let R be a Principal Ideal Domain. Then an R-module G is t.f.b.s. if and only if $G/T(G) \neq p(G/T(G))$ for every irreducible element p in R.

Corollary

Let G be a finitely generated abelian group. Then the following are equivalent.

(1) *G is t.f.b.s.*

(2) *G is t.i.b.s.*

(3) $T(G) \neq G$.

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References

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